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1997 J. Phys. A: Math. Gen. 30 4963

(http://iopscience.iop.org/0305-4470/30/14/010)

J. Phys. A: Math. Gen. **30** (1997) 4963–4982. Printed in the UK PII: S0305-4470(97)81380-7

Universal amplitude ratios in the three-dimensional Ising model

M Caselle†§ and M Hasenbusch‡‖

† Dipartimento di Fisica Teorica dell'Università di Torino, Istituto Nazionale di Fisica Nucleare, Sezione di Torino via P Giuria 1, I-10125 Torino, Italy

‡ Humboldt Universität zu Berlin, Institut für Physik, Invalidenstrasse 110, D-10099 Berlin, Germany

Received 28 January 1997, in final form 12 March 1997

Abstract. We present a high precision Monte Carlo study of various universal amplitude ratios of the three-dimensional Ising spin model. Using state of the art simulation techniques we study the model close to criticality in both phases. Great care is taken to control systematic errors due to finite size effects and correction to scaling terms. We obtain $C_+/C_- = 4.75(3)$, $f_{+,2\mathrm{nd}}/f_{-,2\mathrm{nd}} = 1.95(2)$ and $u^* = 14.3(1)$. Our results are compatible with those obtained by field theoretic methods applied to the ϕ^4 theory and high- and low-temperature series expansions of the Ising model. The mismatch with a previous Monte Carlo study by Ruge *et al* remains to be understood.

1. Introduction

In the neighbourhood of a second-order phase transition various quantities display a singular behaviour. In this limit most of the microscopic features which characterize a given model become irrelevant and models which differ at the microscopic level may share the same singular behaviour. This is the basis of the concept of universality. The first, well known, consequence of universality is that different models belonging to the same universality class share the same critical indices. However, the hypothesis of universality of the various scaling functions has much stronger implications and it is possible to show that models belonging to the same universality class are also characterized by the same values of some critical-point amplitude combinations [1]. Let us see a simple example. Near the critical point the correlation length ξ diverges as

$$\begin{aligned} &\xi \sim f_+ t^{-\nu} \qquad t > 0 \\ &\xi \sim f_-(-t)^{-\nu} \qquad t < 0 \end{aligned} \tag{1}$$

with

$$t = \frac{T - T_c}{T_c} \tag{2}$$

where T is the temperature and T_c is the critical temperature. Different models in the same universality class share not only the same critical exponent ν, but also the same dimensionless combination of critical amplitudes f_+/f_-. This is particularly relevant from

§ E-mail address: caselle@to.infn.it

‖ E-mail address: hasenbus@birke.physik.hu-berlin.de the experimental point of view, since in general critical amplitudes are more easily detectable than critical indices and allow a simpler identification of the universality class. In fact the variations of the critical indices between different universality classes are in general rather small, while the amplitude ratios may vary by large amounts. In this paper we shall be, in particular, interested in the universality class of the three-dimensional Ising model which has several interesting experimental realizations, ranging from the binary mixtures to the liquid vapor transitions.

The two standard approaches to the evaluation of these amplitudes ratios in the Ising case are the use of field theoretic methods applied to the ϕ^4 theory [2–8], and the extrapolation to criticality of low- and high-temperature series expansions on various lattices [9]. All these estimates are in general in rather good agreement among them (for a comparison and a discussion see section 5).

In order to obtain results from Monte Carlo simulations relevant for the scaling limit we have to control both finite size effects as well as corrections to scaling. This means that the linear lattice sizes L have to be chosen such that $L \gg \xi$ while $\xi \to \infty$ as $\beta \to \beta_c$. In practice one has to carefully check which factor of L/ξ is required to obtain results sufficiently close to the thermodynamic limit. While in the high-temperature phase $L/\xi \approx 7$ turns out to be sufficient to give thermodynamic limit results within numerical accuracy, in the low-temperature phase this factor has to be doubled at least. The value of ξ that can be reached, and hence the control of corrections to scaling, is limited by the CPU time available for the study. In the present paper the largest correlation length is $\xi = 11.884(9)$ in the high-temperature phase and $\xi = 6.208(18)$ in the low-temperature phase.

One should also note that simulations in the low-temperature phase are considerably more difficult than those in the high-temperature phase of the model. In the low-temperature phase conceptual as well as practical problems caused by spontaneous symmetry breaking arise. Furthermore, the determination of the correlation length is complicated by the occurrence of secondary correlation lengths which are close to the leading one.

Due to all these reasons it is only recently that there have been some attempts to measure these ratios in Monte Carlo simulations [11]. However, the results are rather puzzling. For instance, in the case of the ratio f_+/f_- discussed above, the Monte Carlo estimate, which is $f_+/f_- = 2.06(1)$ [11], disagrees with the one obtained with strong/weak coupling series $f_+/f_- = 1.96(1)$ [9], while the field theoretical estimate $f_+/f_- = 2.013(28)$ [8] lies in between the two.

The aim of our work is to show that Monte Carlo estimates of the amplitude ratios can indeed be competitive with other approaches. To this end we have elaborated a technique to directly extract the various amplitude ratios, without evaluating the single amplitudes thus avoiding all the uncertainties related to the critical indices. We shall discuss this point in section 5 below. Besides this, we have devoted great care throughout the paper to keep systematic errors due to finite size effects and corrections to scaling under control. Finally we have used state of art simulation techniques to obtain high precision estimates of the observables near the critical point, in both phases. The simulations in the high-temperature phase have been performed using Wolff's single cluster algorithm. Here the improved estimators give a great boost to the accuracy of the results. However, in the low-temperature phase, due to the finite magnetization, the improved estimators of the cluster algorithm are of little help. Hence, we simulate here with a multispin coding implemented Metropolis-like algorithm.

As we shall see, our results are comparable in precision and agree with the most recent field-theoretic and strong/weak coupling estimates, while they are incompatible with the

MC results of Ruge *et al* [11]. A more detailed comparison of the data might be helpful to understand this discrepancy.

This paper is organized as follows. In section 2 we collect some information on the three-dimensional Ising model and on the observables that we study. In section 3 we discuss the details of the simulation, while in section 4 we analyse the scaling behaviour of the measured quantities: magnetization, susceptibility and correlation lengths. In section 5 we study the amplitude ratios and compare our results with other existing estimates and experiments. Finally, section 6 is devoted to some concluding remarks.

2. General setting

2.1. The model

We study the Ising spin model in three dimensions on a simple cubic lattice. The action is given by

$$S_{\text{spin}} = -\beta \sum_{\langle n,m \rangle} s_n s_m \tag{3}$$

where the field variable s_n takes the values -1 and $+1$; $n \equiv (n_0, n_1, n_2)$ labels the sites of the lattice and the notation $\langle n, m \rangle$ indicates that the sum is taken on nearest-neighbour sites only. The coupling β is defined as $\beta \equiv \frac{1}{kT}$, hence the reduced temperature t can be written as

$$t = \frac{\beta_c - \beta}{\beta} \tag{4}$$

where $\beta_c \equiv \frac{1}{kT_c}$. In the following we shall consider n_1 and n_2 as 'space' directions and n_0 as the 'time' direction and shall sometimes denote the time coordinate n_0 with τ. We always consider lattices of equal extension L and periodic boundary conditions in all three directions.

2.2. The observables

2.2.1. Magnetization. The magnetization of a given configuration is defined as:

$$m = \frac{1}{V} \sum_i s_i \tag{5}$$

where $V \equiv L^3$ is the volume of the lattice. However, in a finite volume the Z_2 symmetry of the model cannot be broken for any nonzero temperature. Hence, the expectation value of m vanishes.

In order to obtain the magnetization of the model in the low-temperature phase one should add a magnetic field h in order to break the symmetry. Then one should first take the thermodynamic limit at finite magnetic field and then take the limit of the vanishing magnetic field. However, it is difficult to follow this route in a numerical study.

As an alternative Binder and Rauch [12] suggested simulating the finite lattices at vanishing external field and studing the quantity

$$\langle m \rangle \equiv \lim_{L \to \infty} \sqrt{\langle m^2 \rangle}. \tag{6}$$

However, it turns out that this is not the best choice. In fact this observable is affected by strong finite size effects [13] which would require very large lattices to obtain reliable estimates of the infinite volume magnetization. It has recently been observed [14] that a much more stable observable is:

$$\langle m \rangle \equiv \lim_{L\to\infty} \langle |m| \rangle. \tag{7}$$

The finite size behaviour of this observable, was carefully studied in [14] where it was shown that the asymptotic, infinite volume, value is reached for lattices of size $L >\sim 8\xi$, where ξ denotes the correlation length. In our simulations we always use lattice sizes much larger than this threshold.

Close to the critical temperature, the magnetization is supposed to scale as

$$\langle m \rangle \sim B(-t)^{\beta} \qquad t < 0 \tag{8}$$

where the critical exponent β should not be confused with the inverse temperature.

2.2.2. Magnetic susceptibility. The susceptibility gives the response of the magnetization to an external magnetic field

$$\chi = \frac{\partial \langle m \rangle}{\partial H}. \tag{9}$$

One easily derives that the magnetic susceptibility can be expressed in terms of moments of the magnetization as follows

$$\chi = V(\langle m^2 \rangle - \langle m^2 \rangle). \tag{10}$$

Close to the critical temperature the magnetic susceptibility is supposed to scale as

$$\begin{aligned} \chi &\sim C_+ t^{-\gamma} \qquad t > 0 \\ \chi &\sim C_-(-t)^{-\gamma} \qquad t < 0. \end{aligned} \tag{11}$$

2.2.3. Exponential correlation length. We consider the decay of so-called time-slice correlation functions. The magnetization of a time slice is given by

$$S_{n_0} = \frac{1}{L^2} \sum_{n_1,n_2} s_{(n_0,n_1,n_2)}. \tag{12}$$

Let us define the correlation function

$$G(\tau) = \sum_{n_0} \{\langle S_{n_0} S_{n_0+\tau} \rangle - \langle S_{n_0} \rangle^2\}. \tag{13}$$

The large distance behaviour of $G(\tau)$ is given by

$$G(\tau) \propto \exp(-\tau/\xi) \tag{14}$$

where ξ is the exponential correlation length.

Close to criticality the behaviour of the correlation length is governed by the scaling laws

$$\begin{aligned} \xi &\sim f_+ t^{-\nu} \qquad t > 0 \\ \xi &\sim f_-(-t)^{-\nu} \qquad t < 0. \end{aligned} \tag{15}$$

2.2.4. Second moment correlation length. The square of the second moment correlation length is defined for a generic value of the spacetime dimensions d by

$$\xi_{2\mathrm{nd}}^2 = \frac{\mu_2}{2d\mu_0} \tag{16}$$

where

$$\mu_0 = \lim_{L\to\infty} \frac{1}{V} \sum_{m,n} \langle s_m s_n \rangle_c \tag{17}$$

and

$$\mu_2 = \lim_{L\to\infty} \frac{1}{V} \sum_{m,n} (m-n)^2 \langle s_m s_n \rangle_c. \tag{18}$$

The connected part of the correlation function is given by

$$\langle s_m s_n \rangle_c = \langle s_m s_n \rangle - \langle s_m \rangle^2. \tag{19}$$

This estimator for the correlation length is very popular since its numerical evaluation (say in Monte Carlo simulations) is simpler than that of the exponential correlation length. Moreover, it is the length scale which is directly observed in scattering experiments. However, it is important to stress that it is not exactly equivalent to the exponential correlation length. The relation between the two can be obtained as follows. Let us write

$$\begin{aligned}\mu_2 &= \frac{1}{V} \sum_{m;n} (n-m)^2 \langle s_m s_n \rangle_c \\ &= \frac{1}{V} \sum_{n;m} \sum_{\mu=0}^{d-1} (n_\mu - m_\mu)^2 \langle s_m s_n \rangle_c \\ &= \frac{d}{V} \sum_{n;m} (n_0 - m_0)^2 \langle s_m s_n \rangle_c. \end{aligned}\tag{20}$$

Due to the exponential decay of the correlation function this sum is certainly convergent and we can commute the spatial summation with the summation over configurations so as to obtain

$$\mu_2 = d \sum_{\tau=-\infty}^{\infty} \tau^2 \langle S_0 S_\tau \rangle_c \tag{21}$$

with S_{n_0} given by equation (12). Analogously one obtains

$$\mu_0 = \sum_{\tau=-\infty}^{\infty} \langle S_0 S_\tau \rangle_c. \tag{22}$$

If we now insert these results into equation (16), assume a multiple exponential decay

$$\langle S_0 S_\tau \rangle_c \propto \sum_i c_i \exp(-|\tau|/\xi_i) \tag{23}$$

and replace the summation by an integration over τ we obtain

$$\xi_{2\mathrm{nd}}^2 = \frac{1}{2} \frac{\int_{\tau=0}^{\infty} \mathrm{d}\tau\, \tau^2 \exp(-\tau/\xi)}{\int_{\tau=0}^{\infty} \mathrm{d}\tau\, \exp(-\tau/\xi)} = \frac{\sum_i c_i \xi_i^3}{\sum_i c_i \xi_i} \tag{24}$$

which is equal to ξ^2 if only one state contributes. An interesting consequence of this analysis is that the difference from one of the ratio $\xi/\xi_{2\mathrm{nd}}$ gives an idea of the density of the lowest

Table 1. Results for β_c and for the critical indices given in literature.

Ref.	Method	β_c	γ	ν	β	θ
[15]	FT		1.241(4)	0.630(2)	0.324(6)	0.496(3)
[16]	ϵ-expansion		1.2390(25)	0.6310(15)	0.3270(15)	0.51(3)
[17]	$d = 3$		1.2405(15)	0.6300(15)	0.3250(15)	0.50(2)
[18]	MCRG	0.221 652(3)		0.624(1)		0.50-0.53
[19]	MCRG	0.221 655(1)(1)		0.625(1)		0.44
[20]	MC, FS	0.221 6546(10)	1.237(2)	0.6301(8)	0.3267(10)	0.52(4)
[21]	MC	0.221 6576(22)	1.239(7)	0.6289(8)	0.3258(44)	
[14]	MC	0.221 6544(3)			0.3269(6)	0.508(25)
[22]	HT		1.237(2)	0.6300(15)		0.52(3)
[23]	HT		1.239(3)	0.632(3)		0.52(3)
[24]	HT		1.2385(25)	0.6305(15)		0.57(7)
[25]	HT		1.2395(4)	0.632(1)		0.54(5)
[26]	$d = 2 + 1$, FS		1.236(8)	0.627(4)	0.332(6)	
[27]	$d = 2 + 1$, FS			0.629(2)	0.324(9)	
[28]	$d = 2 + 1$, HT		1.241(3)	0.636(4)		
[29]	$d = 2 + 1$, LT		1.255(10)	0.640(10)	0.320(3)	
[30]	$d = 2 + 1$, FS		1.23(1)	0.627(2)	0.324(3)	

states of the spectrum. If these are well separated the ratio will be almost 1, while a ratio significantly higher than 1 would indicate a denser distribution of states.

The critical behaviour of $\xi_{2\mathrm{nd}}$ is governed by the same critical index ν, so near the critical point we expect:

$$\begin{aligned} \xi_{2\mathrm{nd}} &\sim f_{+,2\mathrm{nd}} t^{-\nu} \qquad t > 0 \\ \xi_{2\mathrm{nd}} &\sim f_{-,2\mathrm{nd}} (-t)^{-\nu} \qquad t < 0. \end{aligned} \tag{25}$$

2.3. Critical indices

Our aim is to obtain high precision estimates for some amplitude ratios. To this end we need to use as input information the critical temperature β_c and the values of the critical indices† defined above. In table 1 we list some estimates for these quantities, obtained with field theoretical methods, strong coupling series and Monte Carlo simulations. The last five values (denoted by '$d = 2 + 1$') refer to results obtained in the framework of the quantum Hamiltonian formulation of the Ising model.

In general these values are in rather good agreement among themselves, despite the fact that they were obtained with very different methods. As input parameters for our analysis we have decided to choose the following values:

$$\beta_c = 0.221\,6544(3) \qquad \gamma = 1.2390(15) \qquad \nu = 0.6310(15) \tag{26}$$

$$\beta = 0.3270(6) \qquad \theta = 0.51(3) \tag{27}$$

which were obtained by combining the results of [14] and [16] and satisfying the scaling relations. Let us stress, however, that our results are only slightly affected by this choice‡.

† As a matter of fact, for the actual determination of the amplitude ratios we only need to know β_c and the correction to scaling exponent θ. The values of the other critical indices will only be use to compare our results with those obtained with the series expansions. For a discussion of the role of θ in our analysis, see section 5.

‡ The systematic errors due to the uncertainties in the choice of β_c and θ turn out to be much smaller than the statistical fluctuations of our estimates. In any case, both are taken into account in the errors that we quote in our final results.

2.4. Amplitude ratios

In the following we shall be interested in these scaling functions:

$$\Gamma_\chi(t) \equiv \frac{\chi(t)}{\chi(-t)} \qquad \Gamma_\xi(t) \equiv \frac{\xi_{2\mathrm{nd}}(t)}{\xi_{2\mathrm{nd}}(-t)} \qquad (t > 0) \tag{28}$$

$$u(t) \equiv \frac{3\chi(t)}{\xi^3_{2\mathrm{nd}}(t) m^2(t)} \qquad (t < 0) \tag{29}$$

$$\Gamma_c(t) \equiv \frac{\chi(t)}{\xi^3_{2\mathrm{nd}}(t) m^2(-t)} \qquad (t > 0) \tag{30}$$

(note the factor of 3 difference between the definitions of Γ_c and u). While Γ_χ, Γ_ξ and Γ_c mix low- and high-temperature observables, u only contains quantities evaluated in the broken symmetry phase. Γ_c and u are scale invariant thanks to the following scaling (and hyperscaling) relations among the critical exponents:

$$\alpha + 2\beta + \gamma = 2 \qquad d\nu = 2 - \alpha. \tag{31}$$

In particular, u plays the important role of a low-temperature renormalized coupling constant in the study of the ϕ^4 theory directly in $d = 3$.

It is important to note that Γ_c is related to the ratio of two amplitude combinations (in which $A(t)$ denotes the specific heat):

$$R_c \equiv \frac{\chi(t)A(t)}{m^2(-t)} \qquad R_\xi \equiv \xi_{2\mathrm{nd}}(t)A(t)^{1/3} \qquad (t > 0) \tag{32}$$

which have been widely studied in literature since they can be evaluated rather easily in experiments. The relation is: $\Gamma_c = R_c/R_\xi^3$. In the scaling limit these functions are related to the amplitudes defined in equations (8), (11), (15) and (25) as follows:

$$\lim_{t\to 0} \Gamma_\chi(t) = \frac{C_+}{C_-} \qquad \lim_{t\to 0} \Gamma_\xi(t) = \frac{f_{+,2\mathrm{nd}}}{f_{-,2\mathrm{nd}}} \tag{33}$$

$$\lim_{t\to 0} u(t) \equiv u^* = \frac{3C_-}{f^3_{-,2\mathrm{nd}} B^2} \qquad \lim_{t\to 0} \Gamma_c(t) = \frac{C_+}{f^3_{+,2\mathrm{nd}} B^2}. \tag{34}$$

Finally we shall also be interested in evaluating the ratio:

$$\frac{\xi}{\xi_{2\mathrm{nd}}} \tag{35}$$

both above and below the critical point.

2.5. Series expansions

A very powerful approach to the study of the three-dimensional Ising model is represented by the series expansions which lead to estimates for several quantities near the critical point which are competitive with the most precise Monte Carlo simulations. In the following we shall compare our results in the low-temperature phase (which is the one in which simulations are more difficult and results are in general affected by stronger finite size effects) with those obtained with series expansions with the twofold aim of testing the reliability of our simulations and comparing the precision of the two methods. The low-temperature regime is also particularly interesting because recently these series have been extended up to very high orders [31–33]. Some information on these series can be found in table 13. In order to extract from the series the estimates for the observables in which we are interested and to quantify the uncertainty of such estimates we use the so-called 'double biased inhomogeneous differential approximants' (IDA). The technique of IDA is described in [9, 34], to which we refer for notations and further details. Following [9] we use the notation [K/L;M] for the approximants. In order to keep the fluctuations of the results under control, we have chosen to use double biased IDA [9], namely we fix both the critical coupling β_c and the critical index describing the critical behaviour of the observable. As K and L vary we obtain several different IDAs and correspondingly several different estimates of the observable. As we approach the critical point these estimates start to spread out, indicating that we are pushing the series towards its convergence threshold. The final problem is then to extract from this set of values the best estimate and its uncertainty. Our choice in this respect is to neglect those IDAs which fluctuate too wildly and treat the remaining approximants on the same ground. To this end we determine the smallest interval that contains half of the results. The values that we shall quote in the tables below as our best estimates correspond to the centre of this interval, the first number in brackets gives half the size of the interval. The second number in brackets gives the error induced by the error of $\beta_c = 0.221\,6544(3)$ and of the critical index used for biasing. Together they give an idea of the uncertainty of the estimate. As we shall see, in the range of coupling in which we are interested, the uncertainty will always be dominated by the spread of IDAs.

3. The simulations

3.1. Simulations in the low-temperature phase

We simulated the Ising spin model in its low-temperature phase at $\beta = 0.2391$, $0.231\,42$, 0.2275, 0.2260, 0.2240 and $0.223\,11$ using a demon algorithm implemented in the multispin coding technique. A detailed discussion of this algorithm is given in [35]. The update of a single spin takes 46×10^{-9} s on a HP 735 and 21×10^{-9} s on a DEC Alpha 250 workstation for $L = 120$. For $\beta = 0.223\,11$ the integrated autocorrelation time of the magnetization was $\tau_{\mathrm{int}} = 81.(2.)$ in units of sweeps.

We used cubical lattices with periodic boundary conditions and a linear extension of about 20ξ. It should be noted that test runs revealed that in contrast to the high-temperature phase, a linear lattice size of 6ξ is clearly not sufficient to obtain results close to the thermodynamic limit. Some information on the simulations is presented in table 2.

We computed $\langle S_0 S_\tau \rangle$ for all values of τ available on the finite lattice. We evaluated S_0 and S_τ for all three lattice directions and all possible translations. The connected part was then obtained by subtracting the expectation value of the square of the magnetization $\langle (1/V \sum_n s_n)^2 \rangle$.

In order to obtain an estimate for the true correlation length we started from the ansatz

$$G(\tau) \propto \exp(-\tau/\xi) + \exp(-(L-\tau)/\xi) \tag{36}$$

Table 2. Statistics of the runs in the low-temperature phase.

β	L	Measures	Sweeps/measure	Bits
0.239 1	30	40 000	25	64
0.231 42	40	50 000	25	64
0.227 5	50	50 000	25	64
0.226 0	80	50 000	25	64
0.224 0	100	92 000	25	32
0.223 11	120	124 000	25	32

Table 3. Results in the low-temperature phase.

β	L	m	E	ξ_{exp}	ξ_{2nd}	χ
0.239 1	30	0.667 162(20)	0.553 732(17)	1.2851(28)	1.2335(15)	4.178(3)
0.231 42	40	0.570 306(16)	0.478 046(12)	1.8637(45)	1.8045(21)	9.394(4)
0.227 5	50	0.491 676(14)	0.430 364(10)	2.578(7)	2.5114(31)	18.706(10)
0.226 0	80	0.449 984(16)	0.409 609(4)	3.103(7)	3.0340(32)	27.596(11)
0.224 0	100	0.372 490(10)	0.378 612(3)	4.606(13)	4.509(6)	61.348(34)
0.223 11	120	0.320 830(10)	0.362 946(2)	6.208(18)	6.093(9)	112.60(7)

where the last term takes into account the periodicity of the lattice. An effective correlation length $\xi_{\mathrm{eff}}(\tau)$ is then computed by solving the equation above for τ and $\tau+1$. Ignoring the term $\exp(-(L-\tau)/\xi)$, $\xi_{\mathrm{eff}}(\tau)$ takes the form

$$\xi_{\mathrm{eff}}(\tau) = \frac{1}{\ln(G(\tau+1)) - \ln(G(\tau))}. \tag{37}$$

For $\tau > 3\xi$, $\xi_{\mathrm{eff}}(\tau)$ seems to stabilize within error bars. In table 3 the results for $\tau = 3\xi$ are given. However, it is important to stress that there might still be systematic errors due to higher excitations that are of the same magnitude as the statistical error of ξ_{eff}. Therefore a multi-exponential ansatz might be useful. We shall discuss this point further in section 4.4.

We compute the second moment of the correlation function by

$$\mu_2 = \sum_{\tau=1}^{\tau_{\mathrm{max}}} \tau^2 G'(\tau) + \sum_{\tau=\tau_{\mathrm{max}}+1}^{\infty} \tau^2 G'(\tau_{\mathrm{max}}) \exp(-(\tau-\tau_{\mathrm{max}})/\xi_{\mathrm{eff}}(\tau_{\mathrm{max}})) \tag{38}$$

where $G'(\tau) = C_{\mathrm{eff}}(\tau)\exp(-\tau/\xi_{\mathrm{eff}}(\tau))$. Where again $C_{\mathrm{eff}}(\tau)$ and $\xi_{\mathrm{eff}}(\tau)$ are obtained from

$$G(\tau) \propto \exp(-\tau/\xi) + \exp(-(L-\tau)/\xi) \tag{39}$$

inserting τ and $\tau+1$. As for the exponential correlation length we use $\tau_{\mathrm{max}} = 3\xi$ to obtain the data reported in table 3. Note that the systematic error introduced by the finite τ_{max} only affects the second term of equation (38), which is small compared with the first one. Hence, these systematic error can be safely ignored for our choice of τ_{max}. The susceptibility is computed analogously. The results are summarized in table 3.

3.2. Simulations in the high-temperature phase

We simulated the Ising spin model in the high-temperature phase using the single cluster algorithm [36]. The time-slice correlation function was determined using the cluster improved estimator. Finite size effects are less important in this phase and preliminary tests showed that lattice sizes greater than 6ξ were enough to keep them under control. This is clearly visible in the data of table 4 where we report a test at $\beta = 0.219\,31$.

Also the determination of the correlation length in the high-temperature phase turns out to be much easier than in the low-temperature phase. The effective correlation length approaches a plateau quite quickly and the true correlation length is well approximated by ξ_{eff} at a self-consistently chosen distance $\tau \approx \xi_{\mathrm{eff}}$. This behaviour of ξ_{eff} also implies that the difference between the second moment correlation length and the true correlation length is much smaller than in the low-temperature phase. We have chosen the β values in the high-temperature phase such that $\beta_c - \beta = \beta_{\mathrm{low}} - \beta_c$, where β_{low} are the inverse temperature used in the simulations of low-temperature phase. The reason for this choice will be made clear in the following section. The uncertainty of β_c virtually does not affect

Table 4. Test for finite size corrections at $\beta = 0.219\,31$.

β	L	Stat	ξ_{exp}	ξ_{2nd}	E	χ
0.219 31	40	50 000 × 1000	8.701(4)	8.691(5)	0.313 986(24)	303.23(30)
0.219 31	50	50 000 × 1000	8.747(4)	8.741(5)	0.313 870(18)	307.14(31)
0.219 31	60	50 000 × 1000	8.754(6)	8.750(5)	0.313 811(15)	307.79(31)
0.219 31	70	50 000 × 1000	8.758(4)	8.751(5)	0.313 823(14)	307.95(31)
0.219 31	80	50 000 × 1000	8.766(5)	8.760(5)	0.313 849(14)	308.58(31)

Table 5. Results in the high-temperature phase.

β	L	Stat	ξ_{exp}	ξ_{2nd}	E	χ
0.204 21	20	50 000 × 500	2.363(1)	2.346(1)	0.262 928(49)	25.255(16)
0.211 89	30	50 000 × 500	3.477(2)	3.465(2)	0.284 663(35)	52.09(4)
0.215 81	40	50 000 × 500	4.864(3)	4.854(3)	0.298 366(28)	98.90(10)
0.217 31	50	50 000 × 800	5.892(3)	5.885(3)	0.304 493(20)	143.04(12)
0.219 31	80	50 000 × 1000	8.766(5)	8.760(5)	0.313 849(14)	308.58(31)
0.220 20	100	50 000 × 1200	11.884(9)	11.877(7)	0.318 742(11)	557.57(61)

the following analysis. The simulation results are summarized in table 5. Recently [37], Monte Carlo results for the second moment correlation length and the magnetic susceptibility were reported. Interpolation of their results, using the scaling ansatz, to our β-values leads to results consistent with ours. One has to note however, that our statistical errors are at least 3 times smaller than those of [37] in the common β-range.

4. Analysis of the results

4.1. Magnetization

A very precise Monte Carlo study of the behaviour of the magnetization in the Ising model can be found in [14]. In particular, in [14] it was shown that the magnetization values obtained from the Monte Carlo simulations were well described by the following empirical approximation:

$$m(\beta) = t^{0.326\,941\,09}(1.691\,904\,5 - 0.343\,577\,31 t^{0.508\,420\,26} - 0.425\,723\,66 t) \qquad (40)$$

with t given by equation (4). Since our values of β are inside the region of validity of this approximation, it is interesting to also compare our magnetization values with equation (40). Note as a side remark, that our estimates for the magnetization are in general more precise than those reported in [14] (where, however, a much larger number of β values was studied). This comparison is reported in table 6, together with the estimates obtained with a double biased IDA analysis of the series published in [33].

It is interesting to note that our values are always in perfect agreement with those obtained from the series expansions, and that our results become more precise than the strong coupling ones starting from $\xi \sim 2$.

The agreement with the data of [14] is also very good. Even if our values are systematically slightly higher than those of equation (40), they are well inside the error bars reported in table 1 of [14].

Table 6. Comparison of our Monte Carlo results for the magnetization with equation (40) and with double biased IDAs.

β	Our MC	Equation (10) of [14]	Biased IDA
0.239 1	0.667 16(2)	0.667 143	0.667 151(3)(1)
0.231 42	0.570 306(16)	0.570 279	0.570 300(16)(1)
0.227 5	0.491 676(14)	0.491 645	0.491 67(4)(1)
0.226 0	0.449 984(16)	0.449 953	0.449 99(7)(1)
0.224 0	0.372 490(10)	0.372 471	0.372 53(15)(2)
0.223 11	0.320 830(10)	0.320 809	0.320 9(2)(1)

Table 7. Comparison of our Monte Carlo results for the susceptibility with double biased IDAs.

β	Our MC	Biased IDA
0.239 1	4.178(3)	4.1801(16)
0.231 42	9.394(4)	9.401(20)
0.227 5	18.706(10)	18.76(15)
0.226 0	27.596(11)	27.67(40)
0.224 0	61.348(34)	61.8(2.7)
0.223 11	112.60(7)	114.6(10.5)

Table 8. Comparison of our Monte Carlo results for the second moment correlation length with double biased IDAs.

β	Our MC	MC of [38]	Biased IDA
0.239 1	1.2335(15)		1.2358(16)
0.231 42	1.8045(21)		1.803(5)
0.227 5	2.5114(31)		2.509(11)
0.226 0	3.0340(32)	3.22(1)	3.034(16)
0.224 0	4.509(6)	4.61(6)	4.493(30)
0.223 11	6.093(9)		6.084(46)

4.2. *Susceptibility*

In table 7 we report the comparison of our data on the susceptibility in the low-temperature phase with a double biased IDA analysis of the series published in [33]. The agreement is again very good and as in the previous case our results become more precise than the strong coupling ones starting from $\xi \sim 2$.

4.3. *Second moment correlation length*

In table 8 we report the comparison of our data on the susceptibility in the low-temperature phase with a double biased IDA analysis of the series published in [31]. Also in this case the agreement is very good. In addition, we give Monte Carlo results of [38] for which the β-value matches ours. One has to note that the results of [38] were obtained with $L \approx 4.7\xi$ and $L \approx 7.8\xi$ for $\beta = 0.226$ and $\beta = 0.224$ respectively.

Table 9. Comparison of the results for the exponential correlation length with those obtained for the Z_2 gauge model ($\tilde{\beta}$ denotes the dual of β).

$\tilde{\beta}$	β	ξ_{gauge}	ξ_{eff}
0.724 84	0.239 10	1.296(3)	1.2851(28)
0.740 57	0.231 42	1.864(5)	1.8637(45)
0.748 83	0.227 50	2.592(5)	2.578(7)
0.752 02	0.226 00	3.135(9)	3.103(7)
0.756 32	0.224 00	4.64(3)	4.606(13)

4.4. *Exponential correlation length*

As we mentioned above, in the low-temperature phase the evaluation of the exponential correlation length is much more delicate than in the high-temperature phase. In particular, we know from the fact that the ratio $\frac{\xi}{\xi_{2\text{nd}}}$ is significantly different from 1 and [39] that in this region the spectrum is very rich and that nearby states exist that could contaminate the measure of ξ. This is exactly the situation discussed in section 3.1 and accordingly we may expect some systematic error in ξ_{eff}. Since the presence of nearby masses is a rather common situation in the broken symmetry phases of statistical mechanical models and since, notwithstanding this, the estimator ξ_{eff} is commonly also used in this case, we have decided to devote this section to a detailed analysis of this problem. We can explicitly see that ξ_{eff} evaluated according to equation (37) is not a good estimator of the true correlation length by comparing our estimates with those of [39] (see table 9). In [39] we computed the glueball spectrum of the Z_2 gauge theory in three dimensions. In $d = 3$ the spin and gauge Ising models are related by duality and the inverse of the 0^+ glueball mass exactly coincides with exponential correlation length of the spin Ising model. In [39] we used a variational approach, using 27 different wilson loops as operators, to obtain a faster convergence of ξ_{eff}. In this way each mass of the spectrum was driven in a different channel, and practically no contamination from higher states was present. The results of [39] are comparable in statistical accuracy with those presented here. It is easy to see looking at table 9 that the values of ξ_{eff} obtained here are systematically smaller, on average by a factor of 0.995(2). This shows, as expected, that the single exponential ansatz is problematic in this case and that a multimass ansatz is needed. In order to have an independent test of this fact we tried to fit our data at $\beta = 0.224$ for the correlation function with the following three-mass ansatz

$$G(\tau) \sim c_1 \exp(-\tau/\xi_1) + c_2 \exp(-\tau/\xi_2) + c_3 \exp(-\tau/\xi_3). \tag{41}$$

However, the main problem of such multimass fits is that they are in general rather unstable under variation of the fit range. This was also the case with our fits. Therefore we fixed the values of $\xi_2 = 2.50$ and $\xi_3 = 1.70$ found in [39]. We found the following values: $c_2/c_1 \approx 0.1$ and $c_3/c_1 \approx 0.04$, when τ's in the range of 5–25 are included in the fit. Note, however, that even in this case the results were still rather unstable, and for this reason we cannot give reliable error bars for our estimates. Assuming that $G(\tau)$ is well described by the three mass-ansatz and using our estimates for c_2/c_1 and c_3/c_1 we obtain $\xi_{\text{eff}}(3\xi) = 0.993\xi$ which is indeed consistent with our result above.

It is also interesting to insert our estimate for c_2/c_1 and c_3/c_1 into equation (24). We obtain $\xi/\xi_{2\text{nd}} = 1.024$.

The situation is much simpler in the high-temperature phase, where $\xi_{2\text{nd}} \sim \xi$, no nearby masses are present and ξ_{eff} is a good estimator of the true correlation length.

Table 10. The various ratios as functions of $\Delta\beta$.

$\Delta\beta$	Γ_χ	Γ_ξ	u	Γ_c	$\frac{\xi}{\xi_{2nd}}$
0.017 45	6.044(5)	1.902(2)	15.00(6)	4.394(9)	1.042(3)
0.009 77	5.546(5)	1.920(3)	14.75(5)	3.850(10)	1.033(3)
0.005 85	5.283(3)	1.932(3)	14.66(6)	3.577(10)	1.026(4)
0.004 35	5.182(5)(1)	1.939(3)	14.64(5)	3.466(8)	1.0227(34)
0.002 35	5.027(6)(2)	1.942(3)	14.47(6)	3.308(9)	1.0215(42)
0.001 46	4.947(6)(3)	1.948(3)(1)	14.51(7)	3.233(9)(1)	1.0188(45)

5. Universal amplitude ratios

The standard approach to evaluate the amplitude ratios is to fit the data obtained for both phases separately with the expected scaling law, and then take the ratio of the amplitudes obtained from the fits.

However, the bias introduced by the uncertainty in the critical exponent can be avoided by directly studying the ratio as a function of the reduced temperature. This is the reason why we carefully chose the couplings so as to have the same differences $\Delta\beta \equiv |\beta - \beta_c|$ in the two phases. To explain our approach better let us study as an example the amplitude ratio Γ_χ. From the data reported in tables 3 and 5 we can compute the ratios of susceptibilities in the low- and high-temperature phase as a function of $\Delta\beta$.

$$\Gamma_\chi(\Delta\beta) = \frac{\chi(\beta_c - \Delta\beta)}{\chi(\beta_c + \Delta\beta)}. \tag{42}$$

As $\Delta\beta$ goes to 0 we expect $\Gamma_\chi(\Delta\beta)$ to converge to the amplitude ratio C_+/C_-. However, the approach to this critical value is rather non-trivial. A naive implementation of the scaling hypothesis would suggest that the data thus obtained should be constant within the error, but it is easy to see, by looking at the data in table 10 that this is not the case. There are in fact two sources of corrections. The fact that observables in both phases are involved in the ratio tells us that we must expect a correction proportional to $\Delta\beta$. Moreover we certainly expect a 'correction to scaling' contribution proportional to $\Delta\beta^\theta$. In the example of Γ_χ the need of such corrections is clearly evident. Looking at the data in table 10 we see that the violations of scaling are much larger than our statistical errors. Even for the smallest values of $\Delta\beta$ we see no stabilization of Γ_χ within error bars. Following the above discussion we fit the data of table 10 with the law

$$\Gamma_\chi(\Delta\beta) = C_+/C_- + a_0\Delta\beta^\theta + a_1\Delta\beta \tag{43}$$

where we assumed that there is no other correction to scaling exponent θ' between θ and 1. The results of these fits for the various ratios in which we are interested are reported in table 11. The fact that we always find rather low $\chi^2_{\rm red}$ strongly supports the correctness of the above assumption.

A few comments are in order at this point.

(1) When dealing with combinations of observables all in the same phase we do not need a correction to scaling term proportional to $\Delta\beta$. This is the case of the coupling

Table 11. Results of the fits according to equations (43) and (44) (see also comment (5)).

	Γ_χ	Γ_ξ	u	Γ_c	$\frac{\xi}{\xi_{2nd}}$
χ^2_{red}	0.52	0.43	0.52	0.01	0.20
CL	59%(2)	65%	67%	98%	90%
a_0	3.5(8)	−0.04(50)	4.8(1.2)	3.0(1.5)	0.24(8)
a_1	46.9(5.4)	−2.9(3.4)		54(10)	
Final result	4.75(3)	1.95(2)	14.3(1)	3.05(5)	1.009(5)

constant u and of the ratio ξ/ξ_{2nd}. In this case we fit the law†:

$$u(\Delta\beta) = u^* + a_0 \Delta\beta^\theta. \tag{44}$$

(2) The error due to the specific choice of β_c is always very small. We report its value in the data of table 10 only when it is not negligible. In these cases the number in the first bracket gives the statistical errors of our data, while the second takes into account the uncertainty of the inverse critical temperature.

(3) In the last row of table 11 we report our final results for the various ratios. The corresponding errors also take into account the uncertainty in the index θ and are thus slightly larger than those extracted by the fits.

(4) In the results of table 11 we always fitted only the last five values of $\Delta\beta$, and systematically discarded the data at $\Delta\beta = 0.017\,45$‡.

(5) Particular care must be devoted to the study of the ratio $\frac{\xi}{\xi_{2nd}}$. It is possible to prove that if higher masses exist in the theory (and this is the case in both phases of the Ising model) then $\frac{\xi}{\xi_{2nd}}$ must certainly be larger than 1. This is a consequence of equation (24) and of the fact that the coefficients c_i which appear in it must be positive (see equations (9) and (10) of [39] for a proof of this last statement).

In the high-temperature phase $\frac{\xi}{\xi_{2nd}}$ is almost compatible with 1 (within the errors), and we can only use our data to set an upper bound for its value which, looking at the data with the largest correlation length, can be safely chosen to be $\frac{f_+}{f_{+,2nd}} < 1.0006$.

In contrast in the $\beta > \beta_c$ phase, the quantity $\frac{\xi}{\xi_{2nd}} - 1$ is much larger than the error bars, and can be measured rather precisely. The data in the last column of tables 10 and 11 refer to this case and use ξ_{eff} (defined in section 4.4) as an estimator of ξ. Hence, we must add to the result of the fit (which is reported in the last row of table 11: $\frac{f_-}{f_{-,2nd}} = 1.009(5)$) the contribution due to the systematic underestimation $\Delta\xi \sim 0.007$ discussed in section 4.4 above. Taking into account this correction as well we quote as our final result $\frac{f_-}{f_{-,2nd}} = 1.017(7)$. It is interesting to note that this result agrees within the errors with the value $\frac{f_-}{f_{-,2nd}} \sim 1.024$ obtained in section 4.4 by inserting into equation (24)

† Note, however, that hidden in the $\Delta\beta$ correction there should also be a term proportional to $\Delta\beta^{2\theta}$ which, due to the fact that $\theta \sim \frac{1}{2}$ is essentially indistinguishable from $\Delta\beta$. The correction $\Delta\beta^{2\theta}$ should also be present in the case in which all the observables belong to the same phase, thus suggesting also using in this case the fit equation (43). It turns out, however, that such a $\Delta\beta^{2\theta}$ correction, if present, has a negligible amplitude and for this reason we confined ourselves to the fit (44).

‡ This is not due to the fact that by adding this data we had a poorer fit, in contrast we checked that for all the ratios the fit keeping *all the six* data was always equally good. The reason for our choice is that we tried to confine ourselves to the narrowest possible region near the critical point compatible with a reasonable precision for the results. This allows us to trust in our assumption of neglecting other possible, unknown, corrections to scaling which are certainly present but hopefully negligible in this range. It is a remarkable consequence of the high precision of our Monte Carlo estimates that we can still extract meaningful results by using only five values of $\Delta\beta$.

our estimates for c_2/c_1 and c_3/c_1 and the values of the two nearby (inverse) masses ξ_2 and ξ_3 extracted from [39]. Finally we can directly estimate the ratio by using the unbiased data for ξ obtained in [39] and reported in table 9. The only problem is that these data are slightly less precise and that the lowest value of β is missing. The resulting estimate for the ratio: $\frac{f_-}{f_{-,2\mathrm{nd}}} = 1.029(11)$ is thus affected by a larger error. Also, in this case we find a good agreement within the errors with our final result $\frac{f_-}{f_{-,2\mathrm{nd}}} = 1.017(7)$

5.1. Comparison with other existing estimates

In table 14 we compare our results with those obtained with other methods. Let us briefly comment on this comparison.

(1) There are three possible approaches to the evaluation of the amplitude ratios. Monte Carlo simulations (denoted 'MC' in table 14), low- and high-temperature series expansions (denoted 'HT, LT', in table 14) and field theoretic methods. In this last case two different approaches are possible. The first one consists of looking at the ϵ expansion of the ϕ^4 theory around four dimensions ('$\epsilon - \exp .$' in table 14). The second one consists of looking directly at the ϕ^4 theory in three dimensions ('d=3' in table 14). For a detailed discussion of these approaches see, for instance [40]. Let us also mention for completeness that an independent, interesting method of evaluating the ratio of specific heat amplitudes (which we do not study in this paper) by looking at the distribution of the zeros of the partition function was proposed and applied to the Ising model in [41, 42].

(2) The results of [2, 4] for Γ_χ and Γ_ξ were obtained with a careful resummation of two loop ϵ-expansions. In contrast, the ϵ-expansion for the exponential ξ (which is needed to obtain the value of $f_-/f_{-,2\mathrm{nd}}$ which is reported in the last column of table 14) is known only at one loop, hence the value $f_-/f_{-,2\mathrm{nd}} \sim 1.005$ of [2] must only be considered as indicative. Later the ϵ-expansion for Γ_χ was extended up to ϵ^3 and the value of [4] corresponds to the Padé resummation of such series. Recently [5], this result was further improved by using the parametric representation of the equation of state of the theory.

(3) The $d = 3$ approach originates from a suggestion by Parisi [17]. While the results of [5, 6] use only series expansions obtained in the symmetric phase of the theory, in [7, 8] a three loop calculation, directly performed in the low-temperature phase, was used. In this last case a crucial role is played by the low-temperature renormalized coupling constant u evaluated at the critical point. We shall further comment on this point later.

(4) The estimates for the amplitude ratios obtained by using the low- and high-temperature series expansion reported in table 14 are mainly taken from [9]. They were obtained by using IDAs on the series reported in table 12, where we used the standard notations: $v \equiv th(\beta)$ and $u \equiv \mathrm{e}^{-4\beta}$ (not to be confused with the low-temperature coupling constant in the $d = 3$ ϕ^4 theory!) for high- and low-temperature series respectively.

Table 12. Some information on the series used in [9].

Ref.	Year	Observable	Length
[43]	1979	HT/χ	v^{18}
[44]	1969	$\mathrm{HT}/\xi_{2\mathrm{nd}}$	v^{12}
[46]	1979	LT/χ	u^{20}
[45]	1975	$\mathrm{LT}/\xi_{2\mathrm{nd}}$	u^{15}
[45]	1975	LT/ξ	u^7
[46]	1979	LT/m	u^{21}

Table 13. Some information on the low-temperature series used in this paper.

Ref.	Year	Observable	Length
[33]	1993	LT/χ	u^{32}
[31]	1995	$\mathrm{LT}/\xi_{2\mathrm{nd}}$	u^{23}
[32]	1995	LT/ξ	u^{15}
[33]	1993	LT/m	u^{32}

Table 14. Results for the amplitude ratios reported in literature.

Ref.	Year	Method	$\frac{C_+}{C_-}$	$\frac{f_{+,2\mathrm{nd}}}{f_{-,2\mathrm{nd}}}$	u^*	$\frac{C_+}{f^3_{+,2\mathrm{nd}}B^2}$	$\frac{f_-}{f_{-,2\mathrm{nd}}}$
[2, 3]	1974	ϵ-exp	$\sim$4.8	$\sim$1.91			$\sim$1.005
[4]	1985	ϵ-exp	$\sim$4.9				
[5]	1996	ϵ-exp	4.70(10)				
[6]	1987	$d = 3$	4.77(30)			$\underline{3.02(8)}$	
[47]	1993	$d = 3$					$\sim$1.0065
[5]	1996	$d = 3$	4.82(10)				
[8]	1996	$d = 3$	4.72(17)	2.013(28)	$\underline{14.4(2)}$		
[45]	1975	HT,LT					$\sim$1.0069
[9]	1989	HT,LT	4.95(15)	1.96(1)	$\underline{14.8(1.0)}$	$\underline{3.09(8)}$	
[48]	1993	HT,LT			14.73(14)		
[11]	1994	MC	$\underline{5.18(33)}$	2.06(1)	$\underline{17.1(1.9)}$	$\underline{3.36(23)}$	
this work	1996	MC	4.75(3)	1.95(2)	14.3(1)	3.05(5)	1.017(7)

Recently these data were reanalysed [10] (using the same series) leading to essentially the same results. Note, however, that recently new, longer series have been constructed in the low-temperature phase. These are the series that we have used in the previous sections to test our Monte Carlo results. It would be very interesting to see whether these new series can lead to improved estimates of the amplitude ratios. In particular it would now be possible to analyse also the ratio $\xi/\xi_{2\mathrm{nd}}$ which was previously almost unaccessible, since the two series for ξ and $\xi_{2\mathrm{nd}}$ start to be different only at the order u^6. In fact the estimate for this ratio reported in table 14 (taken from [45]) was obtained by using only one (the last) element of the LT series for ξ and is thus rather unreliable.

(5) Some of the data reported in table 14 (those which are underlined) were obtained by combining separate amplitudes reported by the authors, thus their errors are most probably overestimated.

(6) An estimate for the ratio $\frac{f_+}{f_{+,2\mathrm{nd}}}$ has recently been reported [49], obtained with a strong coupling expansion to 15th order of the correlation function $G(\tau)$. The result is $\frac{f_+}{f_{+,2\mathrm{nd}}} = 1.000\,23(5)$ which agrees with our bounds $1.0 < \frac{f_+}{f_{+,2\mathrm{nd}}} < 1.0006$.

(7) Finally, it is also interesting to compare our results with those obtained in the framework of the mean-field approximation, which gives $\frac{C_+}{C_-} = 2$ and $\frac{f_+}{f_-} = \sqrt{2}$ (see [1] for details).

5.2. Comparison with an effective potential model

It has recently been proposed to study the critical properties of the three-dimensional Ising model by constructing the effective potential of the corresponding quantum field theory. This effective potential is constructed by simulating the model for various values of the

Table 15. Comparison with Tsypin's results.

Observable	This work	Table 1 of [51]	Table 2 of [51]
m	0.449 984(16)	0.449 75(17)	0.449 75(17)
χ	27.596(11)	27.397(75)	27.586(198)
$\xi_{2\text{nd}}$	3.034 0(32)	2.946(6)	2.956(10)
u	14.64(5)	15.90(9)	15.84(28)
L	80	30	30

Table 16. Experimental estimates for some amplitude ratios.

Experimental set-up	$\frac{C_+}{C_-}$	$\frac{f_{+,2\text{nd}}}{f_{-,2\text{nd}}}$	$\frac{C_+}{f_{+,2\text{nd}}^3 B^2}$
(bm)	4.4(4)	1.93(7)	3.01(50)
(lvt)	4.9(2)		2.83(31)
(af)	5.1(6)	1.92(15)	
(all of them)	4.86(46)	1.93(12)	2.93(41)
this work	4.75(3)	1.95(2)	3.05(5)

external magnetic field. This programme was carried out in [50] for the high-temperature phase of the model and was recently extended to the broken symmetric phase in [51]. The main result is that in the effective potential, besides the expected ϕ^4 term, a ϕ^6 term is also present. It is interesting to test this model with our high-precision results. Fortunately one of the values of β studied in [51]: $\beta = 0.2260$ exactly coincide with one of our values thus allowing a detailed comparison. This comparison is reported in table 15, where in the third column we have reported the values of the various observables directly measured at $\beta = 0.2260$ (table 1 of [51]) while in the last column we have reported the same observables obtained from what was considered in [51] as the most successful fitting procedure (table 2 of [51]). Note that χ is the inverse of V'' and that $u = 3G$ in the notations of [51]. Our results are in general one order of magnitude more precise than those of [51]. It is interesting to see that both m and χ are in rather good agreement with our data. The only strong disagreement is in the value of $\xi_{2\text{nd}}$ and, as a consequence of this, in u. This disagreement is most likely only due to the too small lattices studied in [51] (the lattice size is reported in the last line of table 15) and does not imply that the approach proposed in [51] is wrong.

5.3. Comparison with experimental data

The experimental data reported in table 16 refer to the three most important experimental realizations of the Ising universality class, namely binary mixtures (bm), liquid–vapour transitions (lvt) and uniaxial antiferromagnetic systems (af). It is important to note that these realizations are not on the same ground. Af systems are particularly apt to measure the C_+/C_- and $f_{+,2\text{nd}}/f_{-,2\text{nd}}$ ratios, while for the lvt the $\Gamma_c \equiv R_c/R_\xi^3$ combination is more easily accessible. Finally, in the case of bm all the three ratios can be rather easily evaluated. Even if obtained with very different experimental set-ups all these estimates qualitatively agree among themselves and this is certainly one of the most remarkable experimental evidences of universality. When looking in more detail at the various results one can see a residual small spread among them (even if in general the various estimates are compatible

Table 17. Experimental estimates for the $\frac{f_{+,2\mathrm{nd}}}{f_{-,2\mathrm{nd}}}$ ratio.

Ref.	Year	Experimental set-up	$\frac{f_{+,2\mathrm{nd}}}{f_{-,2\mathrm{nd}}}$
[52]	1971	(af), MnF_2	1.7(3)
[53]	1972	(af), FeF_2	2.06(20)
[54]	1980	(af), CoF_2	1.93(10)
[55]	1983	(bm), N–H	1.9(2)
[56]	1986	(bm), I–W	2.0(4)
this work	1996	MC	1.95(2)

within the quoted experimental uncertainties). This spread is mainly due to the presence of correction to scaling terms whose amplitudes vary as the experimental realizations are changed and that are difficult to control. Thus some care is needed to compare these experimental data with theoretical estimates. The common attitude is to assume that the above systematic errors are randomly distributed and to take the weighted mean of the various experimental results.

Following this, in table 16 we have reported the weighted means (together with, in parenthesis, the standard deviations), of the experimental results reported in [1]. In the first three rows we have studied separately the three different realizations of the universality class while in the fourth row all the experimental data at disposal are analysed together. In the last row we have reported our results.

In the case of the $f_{+,2\mathrm{nd}}/f_{-,2\mathrm{nd}}$ ratio (for which, as we have seen, some of the present theoretical or Monte Carlo estimates disagree) we have listed, for a more detailed comparison, all the available experimental data in table 17. In this table we denote with ‘N–H’ the nitrobenzene–n-hexane binary mixture, and with ‘I–W’ the one obtained by mixing isobutyric acid and water. A much more detailed account of the various experimental estimates can be found in [1].

6. Conclusions

We have estimated various universal amplitude ratios in the case of the three-dimensional Ising model. Our final results are:

$$\frac{C_+}{C_-} = 4.75(3) \qquad \frac{f_{+,2\mathrm{nd}}}{f_{-,2\mathrm{nd}}} = 1.95(2) \qquad \frac{f_-}{f_{-,2\mathrm{nd}}} = 1.017(7) \tag{45}$$

$$u^* \equiv \frac{3C_-}{f^3_{-,2\mathrm{nd}}B^2} = 14.3(1) \qquad \frac{C_+}{f^3_{+,2\mathrm{nd}}B^2} = 3.05(5). \tag{46}$$

Our results are, in general, in good agreement with other estimates of the same quantities obtained with field theoretical methods or with high/low-temperature series. The main discrepancy that we have found is with the Monte Carlo results of [11] and with some of the results of [51]. It must also be noted that our result $\frac{f_{+,2\mathrm{nd}}}{f_{-,2\mathrm{nd}}} = 1.95(2)$ only marginally agrees with that of [8]: 2.013(28). However, as we mentioned above, this result depends on the value of the coupling constant u^* which is an external input in the calculations of [8]. By plugging our value of u^* into the perturbative expansion of [8] we find a lowering of the ratio with respect to [8]. This lower result: $\frac{f_{+,2\mathrm{nd}}}{f_{-,2\mathrm{nd}}} = 1.99(2)$ agrees not only with the strong/weak coupling result [9], but also with our estimate. Finally, it is important to note that our results are also in reasonable agreement with the experimental ones.

Acknowledgments

We thank F Gliozzi, K Pinn, P Provero and S Vinti for many helpful discussions. This work was partially supported by the European Commission TMR programme ERBFMRX-CT96-0045.

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